Kuratowski Numbers of Finite Topological Spaces

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What is a Topology?

Definition

A *topology* on a set X consists of a set $\mathcal{U}$ of subsets of X, called the "open sets of X in the topology $\mathcal{U}$ ", with the following properties.

- The empty set $\emptyset$ and the set X are in $\mathcal{U}$
- A finite intersection of sets in $\mathcal{U}$ is in $\mathcal{U}$
- An arbitrary union of sets in $\mathcal{U}$ is in $\mathcal{U}$

Closed Sets

Definition

Let X be a topological space. A subset of X is *closed* if its complement is open. The closed sets satisfy the following conditions.

- The empty set $\emptyset$ and the set X are closed.
- An arbitrary intersection of closed sets is closed.
- A finite union of closed sets is closed.

Examples of Topologies

Example

Let $X = \{1, 2, 3, 4\}$
$\mathcal{U} = \{\emptyset, X\}$ Indiscrete Topology

Example

Let $X = \{1, 2, 3, 4\}$
$\mathcal{U} = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ Discrete Topology

Example

Let $X = \{1, 2, 3, 4\}$
$\mathcal{U} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

Separation Axioms

A *neighborhood* of a point $x \in X$ is an open set U such that $x \in U$

Definition

Let $\{X, \mathcal{U}\}$ be a topological space.

- X is a $T_0 -$ *space* if for any two points of X, there is an open neighborhood of one that does not contain the other. That is, the topology distinguishes points.
- X is a $T_1 -$ *space* if each point of X is a closed subset.
- X is a $T_2 -$ *space*, or *Hausdorff* space, if any two points of have disjoint open neighborhoods.

Bases for Topologies

Definition

A *basis* for a topology on a set X is a set $\mathcal{B}$ of subsets of X such that

- For each $x \in X$, there is at least one $B \in \mathcal{B}$ such that $x \in B$
- If $x \in B' \cap B''$ where $B', B'' \in \mathcal{B}$, then there is at least one $B \in \mathcal{B}$ such that $x \in B \subset B' \cap B''$

In a finite topological space, the set of open sets U_x, where U_x is the intersection of the open sets that contain x, is a basis $\mathcal{B}$ for X. If $\mathcal{C}$ is any other basis, then $\mathcal{B} \subset \mathcal{C}$. Therefore $\mathcal{B}$ is a unique minimal basis for X.

Building a Matrix from a Topology

Definition

Consider square matrices $M = (a_{i,j})$ with integer entries that satisfy the following properties.

- $a_{i,i} \geq 1$.
- $a_{i,j}$ is -1,0, or 1 if $i \neq j$
- $a_{i,j} = -a_{j,i}$ if $i \neq j$
- $a_{i_1,i_s} = 0$ if there is a sequence of distinct indices $\{i_1, ..., i_s\}$ such that $s > 2$ and $a_{i_k,i_{k+1}} = 1$ for $1 \leq k \leq s-1$.

For a minimal basis $U_1, ..., U_r$ of a topology $\mathcal{U}$ on a finite set X, define an $r \times r$ matrix $M = (a_{i,j})$ as follows. If $i = j$, let $a_{i,i}$ be the number of elements $x \in X$ such that $U_x = U_i$. Define $a_{i,j} = 1$ and $a_{j,i} = -1$ if $U_i \subset U_j$ and there is no k (other than i or j) such that $U_i \subset U_k \subset U_j$. Define $a_{i,j} = 0$ otherwise.

Matrix Construction

Example

Let $X = \{1, 2, 3, 4\}$
$\mathcal{U} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$
$\mathcal{B}$=$\{\{1\},\{2\},\{2,3\},X\}$ minimal basis

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

To see that our mapping is onto, consider and $r \times r$ matrix M of the sort under consideration and let X be the set of pairs of integers (u, v) with $1 \leq u \leq r$ and $1 \leq v \leq a_{u,u}$. Define subsets U_i of X by letting U_i have elements those $(u, v) \in X$ such that either $u = i$ or $u \neq i$ but $u = i_1$ for some sequence of distinct indices $\{i_1, ..., i_s\}$ such that $s \geq 2$, $a_{i_k, i_{k+1}} = 1$ for $1 \leq k \leq s - 1$, and $i_s = i$. We see that the U_i give a minimal basis for a topology by verifying conditions in the definition for a minimal basis.

Big Theorem

Say that two such matrices M and N are equivalent if there is a permutation matrix T such that $T^{-1}MT = N$ and let $\mathcal{M}$ denote the set of equivalence classes of such matrices.

Theorem

The homeomorphism classes of finite spaces are in bijective correspondence with $\mathcal{M}$. If the homeomorphism class of X corresponds to the equivalence class of an $r \times r$ matrix M, then r is the number of sets in a minimal basis for X, and the trace of M is the number of elements of X. Moreover, X is a T_0- space if and only if the diagonal entries of M are all one.

Kuratowski's 14-Set Theorem

There are at most 14 distinct sets that can be created by taking closures and complements from a subset of a topology.

- Let A be a subset of a topological space X. The *closure* $\bar{A}$ of A is the intersection of the closed sets containing A.

14-Set

Let

$$X = \{1, 2, 3, 4, 5, 6, 7\}$$

and

$$\mathcal{B} = \{\{1\}, \{7\}, \{1, 2\}, \{3, 5\}, \{6, 7\}, X\}$$

a generating basis
for the topology:

$$\begin{aligned}\mathcal{U} =& \{\emptyset, X, \{1\}, \{7\}, \{1, 2\}, \{1, 7\}, \{3, 5\}, \{6, 7\}, \{1, 2, 7\}, \{1, 3, 5\},\\ &\{1, 6, 7\}, \{3, 5, 7\}, \{1, 2, 3, 5\}, \{1, 2, 6, 7\}, \{3, 5, 6, 7\}\}\end{aligned}$$

closed sets of $\mathcal{U}$:

$$\begin{aligned}\bar{\mathcal{U}} =& \{\emptyset, X, \{1, 2, 4\}, \{3, 4, 5\}, \{4, 6, 7\}, \{1, 2, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 6, 7\},\\ &\{3, 4, 5, 6\}, \{1, 2, 3, 4, 5\}, \{1, 2, 4, 6, 7\}, \{2, 3, 4, 5, 6\}, \{3, 4, 5, 6, 7\},\\ &\{1, 2, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6, 7\}\}\end{aligned}$$

14-Set

$$
\begin{aligned}
U_1 &= \{1\} \\
U_2 &= \{1, 2\} \\
U_3 &= \{3, 5\} = U_5 \\
U_4 &= X \\
U_6 &= \{6, 7\} \\
U_7 &= \{7\}
\end{aligned}
$$

$$
M = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 \\
0 & -1 & -1 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

14-Set

$$\begin{aligned}
A &= \{2,5,7\} \\
\bar{A} &= \{2,3,4,5,6,7\} \\
(\bar{A})^c &= \{1\} \\
\overline{((\bar{A})^c)} &= \{1,2,4\} \\
(\overline{((\bar{A})^c)})^c &= \{3,5,6,7\} \\
\overline{((\overline{((\bar{A})^c)})^c)} &= \{3,4,5,6,7\} \\
(\overline{((\overline{((\bar{A})^c)})^c)})^c &= \{1,2\}
\end{aligned}$$

$$\begin{aligned}
A^c &= \{1,3,4,6\} \\
\overline{(A^c)} &= \{1,2,3,4,5,6\} \\
(\overline{(A^c)})^c &= \{7\} \\
\overline{((\overline{(A^c)})^c)} &= \{4,6,7\} \\
(\overline{((\overline{(A^c)})^c)})^c &= \{1,2,3,5\} \\
\overline{((\overline{((\overline{(A^c)})^c)})^c)} &= \{1,2,3,4,5\} \\
(\overline{((\overline{((\overline{(A^c)})^c)})^c)})^c &= \{6,7\}
\end{aligned}$$

Kuratowski 14-Set Proof

To prove Kuratowski's Theorem we define complement, closure, and interior (c, k, and i) operators which satisfy these relations:

$$k^2 = k, c^2 = I, i = ckc, i^2 = i, ic = ck, kc = ci$$

It follows from these relations that any word in k, i, c can be reduced to a form in which c appears either as the leftmost element only, or not at all:

$$(k, c) = (k, i, c) = \{I, i, ik, iki, k, ki, kik, c, ci, cik, ciki, ck, cki, ckik\}$$

By Induction

Base Case: Trivial, every word of length one is one of the 14 words
Inductive Hypothesis: Suppose any word of length n can be reduced to one of the 14 words
Then we can use the IH to write that word $n+1$ by adding a c or a k on the rightmost end.
Case 1 (adding c's)

1. $Ic \to c$
2. $ic \to ck$
3. $ikc \to ici \to cki$
4. $ikic \to ikck \to icik \to ckik$
5. $kc \to ci$
6. $kic \to kck \to cik$
7. $kikc \to kici \to kcki \to ciki$
8. $cc \to I$
9. $cic \to cck \to k$
10. $cikc \to cici \to ccki \to ki$
11. $cikic \to cikck \to cicik \to cckik \to kik$
12. $ckc \to cci \to i$
13. $ckic \to ckck \to ccik \to ik$
14. $ckikc \to ckici \to ckcki \to cciki \to iki$

By Induction

Case 2 (adding k's)

1.$Ik \to k$

2.$ik \to ik$

3.$ikk \to ik$

4.$ikik \to ckckckck \to ckckckci \to ckckccii \to ckcki \to cciki \to iki$

5.$kk \to k$

6.$kik \to kik$

7.$kikk \to kik$

8.$ck \to ck$

9.$cik \to cik$

10.$cikk \to cik$

11.$cikik \to cckckckck \to cckckckci \to cckckccii \to ccciki \to cik$

12.$ckk \to ck$

13.$ckik \to ckik$

14.$ckikk \to ckik$

Therefore, by mathematical induction we can write any word of length n and reduce it to one of the 14 forms $\square$

Project Questions

To figure out how information about topologies are inclosed in their represented matrices. Just by looking at the matrix how would you find:

- The Kuratowski number?
- The closure of a subset?

Our T_0 Example

Let $X = \{1, 2, 3, 4\}$
$\mathcal{B}$=$\{\{1\},\{2\},\{2,3\},X\}$ minimal basis
$\mathcal{U} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$
$\bar{\mathcal{U}} = \{\emptyset, X, \{4\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 1 \end{pmatrix}$$

1.$A = \{1, 3\}$
2.$\bar{A} = \{1, 3, 4\}$
3.$(\bar{A})^c = \{2\}$
$\overline{((\bar{A})^c)} = \{2, 3, 4\}$
$(\overline{((\bar{A})^c)})^c = \{1\}$
$\overline{((\overline{((\bar{A})^c)})^c)} = \{1, 4\}$

4.$A^c = \{2, 4\}$
5.$\overline{(A^c)} = \{2, 3, 4\}$
6.$(\overline{(A^c)})^c = \{1\}$
7.$\overline{((\overline{(A^c)})^c)} = \{1, 4\}$
8.$(\overline{((\overline{(A^c)})^c)})^c = \{2, 3\}$
$\overline{((\overline{((\overline{(A^c)})^c)})^c)} = \{2, 3, 4\}$

The Kuratowski Number

- For a $T_0 - space$ (all 1's on the diagonal) the K number is twice the number of subsets in the closure-complement cycle.
- Be smart about your choosing of a subset because different subsets have different K numbers. To find the largest K number of a topology, you have to choose the subset that has points represented in every level topology diagram.
- For non-$T_0 - space$ (not all 1's on the diagonal), you have to find the K number the long way.

Closure of a subset

- For a $T_0 - space$, look at the columns in which your subset has points. Look up the column for 1's to see if points less than the column number are in the closure. Look down the row for 1's (or 0's that satisfy the $U_i \subset U_k \subset U_j$ condition) to see if points greater than the column number are in the closure.

Bibliography

- J. P. May, *Finite spaces and larger contexts: Chapter 1*.
- David Sherman, *Variations on Kuratowski's 14-set Theorem*.
- H. H. Herda and R. C. Metzler, *Closure and Interior in Finite Topological Spaces*. Colloquium Mathematicum, Vol. XV, 1966.

Topology and Matrix Construction

To construct a matrix from a topology: Ask questions about whether $U_i \subset U_j$

$U_1 = \{1\}$

$U_2 = \{2\}$

$U_3 = \{2, 3\}$

$U_4 = X$

Is $U_1 \subset U_2$? No, so for $a_{1,2}$ write a 0

Is $U_1 \subset U_4$? Yes, so for $a_{1,4}$ write a 1

Is $U_2 \subset U_4$? Yes, but $U_2 \subset U_3 \subset U_4$ so for $a_{2,4}$ write a 0

To construct a topology from a matrix: Find the U_i by looking at the $a_{i,j}$ that are 1, and ask questions about the ordered pairs (u, v)

$X = (1, 1), (2, 1), (3, 1), (4, 1)$

Is (2,1) in U_1? $a_{2,3} = 1 \rightarrow a_{3,1} \neq 1$ so No

Is (2,1) in U_3? $a_{2,3} = 1 \rightarrow a_{3,3} = 1$ so Yes

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 1 \end{pmatrix}$$